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Theory of adiabatic fluctuations: third-order noise

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Abstract. We consider the response of a dynamical system driven by external adiabatic fluctuations. Based on the 'adiabatic following approximation' we have made a systematic separation of timescales to carry out an expansion in $\alpha|\mu|^{-1}$, where α is the strength of fluctuations and $|\mu|$ is the damping rate. We show that the probability distribution functions obey the differential equations of motion which contain third-order terms (beyond the usual Fokker–Planck terms) leading to non-Gaussian noise. The problem of adiabatic fluctuations in velocity space which is the counterpart of Brownian motion for fast fluctuations, has been solved exactly. The characteristic function and the associated probability distribution function are shown to be of stable form. The linear dissipation leads to a steady state which is stable and the variances and higher moments are shown to be finite.

1. Introduction

In this paper we discuss the stochastic dynamics of a system driven by external, adiabatic fluctuations. The opposite counterpart of these processes correspond to stochastic processes with fast fluctuations which are more frequently encountered in physical and chemical sciences. The classic and well known problem of the latter kind is the century-old paradigm of Brownian motion first correctly formulated by Einstein [1, 2]. In dealing with fast stochastic processes one essentially examines the average motion of the system subjected to fast fluctuations (which may be of external or of internal type) with the following separation of timescales in mind. If τ_c is the correlation time of fluctuations which is the shortest timescale in the dynamics, compared with coarse-grained timescale Δt over which one follows the average evolution, then

$$\tau_c \ll \Delta t \ll \frac{1}{|\mu|} \tag{I}$$

where $|\mu|^{-1}$ refers to the inverse of the damping rate (or to the inverse of the largest eigenvalue of the 'unperturbed' system). Herein we analyse the average dynamics of a general multivariate nonlinear system subjected to *external, adiabatically slow* fluctuations. We have derived the equation of motion for evolution of the probability distribution function in phase space on a coarse-grained timescale Δt assuming that Δt satisfies the following inequality

$$\frac{1}{|\mu|} \ll \Delta t \ll \tau_c. \tag{II}$$

The slow fluctuations characterized by very long correlation time have received a lot of attention from various workers over the years [2, 3]. While the treatment of stochastic differential equations with fast fluctuation is based on the assumption that there is a very short correlation time such that one is allowed to make an appropriate expansion in $\alpha\tau_c$, where α is the strength of fluctuation, a simplified assumption for dealing with long correlation time is relatively scarce. In general, the problem of long correlation time is handled at the expense of a severe restriction on the type of stochastic behaviour. For instance, several authors [2, 3] have tried the linear and nonlinear models within the framework of simple Markov processes of the type, dichotomic process, two-state Markov process, random telegraphic process, etc. Our strategy here is to follow a *perturbative approach*, pertaining to the separation of the timescale (II) without *keeping any above-mentioned restriction on the type of stochastic behaviour*. Based on the 'adiabatic following approximation' [4] we have recently [5] carried out an expansion in $\alpha|\mu|^{-1}$ to obtain a linear differential equation for the average solution. In this paper we extend this analysis to treat *nonlinear* stochastic differential equations for the construction of appropriate master equations. The perturbative expansion is essentially a counterpart of expansion in $\alpha\tau_c$ as dealt with in the case of fast fluctuations [2]. The difference between the two expansion schemes lies in the identification of two distinct shortest timescales in the dynamics of the two cases. In the case of fast fluctuations it is τ_c, whereas the corresponding role is played by $|\mu|^{-1}$ in adiabatic fluctuations.

We have shown that the equation of motion in phase space for the probability distribution function contains beyond the ordinary Fokker–Planck terms, third-order derivative terms. As shown by Pawula [6] for the one-dimensional case, an equation with third-order derivative terms is in contradiction with the positivity for the transition probability for short time. However it is well known that finite derivative terms of order larger than two may be quite useful on different occasions [7, 8], e.g. in the treatment of optical bistability described in terms of the quasidistribution function of Wigner in quantum optics [7] or in explaining trimolecular reactions using a Poisson representation of the Fokker–Planck equation, and also in a one-dimensional random walk with a boundary within a scheme of expansion of the master equation [8]. Although at this stage of development a clear general probabilistic interpretation in terms of any real stochastic process is lacking [7] one can identify the noise terms with distinct characteristics for such processes. In a similar spirit we are led to the conclusion in the present context that adiabatic fluctuations give rise to third-order non-Gaussian noise terms beyond the usual Fokker–Planck terms.

The central result of this paper is the solution of the problem of adiabatic fluctuations in velocity space, which is the counterpart of Brownian dynamics for rapid fluctuations. We have shown that the characteristic function obeys a simple third-order differential equation. This can be solved exactly to obtain a probability distribution of a stable form which, for small arguments, displays a power law behaviour. It is also important to note that the linear dissipation leads to a stable steady-state distribution. However, the fluctuation being external the energy supplied by this cannot be balanced by dissipation and as such there is no fluctuation–dissipation relation in this case. Furthermore, the non-Gaussian statistical characteristics can be obtained from the calculation of variances and higher moments which are shown to be finite. We thus conclude that although in many cases third-order noise makes the probabilistic consideration truly difficult, the systems driven by adiabatic fluctuations which display a distinct non-Gaussian stochastic behaviour are amenable to understanding in simple probabilistic terms. Occasionally, wherever possible, we allow ourselves a fair comparison with Levy processes [9, 11] and point out the essential differences.

The outline of the paper is as follows. In section 2 we review the basic aspects of adiabatic fluctuations in linear processes as dealt with in our earlier paper [5]. The two basic assumptions, the adiabatic following approximation and the decoupling approximation as well as validity and convergence of perturbative expansion, were discussed in detail in the earlier paper [5]. To make this paper self-contained and readable we review its salient features. In section 3 we extend the treatment to nonlinear equations. The equations in phase space have been derived in section 4. The counterparts of Brownian motion in velocity space for slow fluctuations have been treated in section 5. Explicit solutions for the probability distribution function and the approach to equilibrium have been discussed. This paper is concluded in section 6.

2. Linear processes with adiabatic fluctuations

To begin with we have considered the following linear equation,

$$\dot{u} = \{\boldsymbol{A}_0 + \alpha \boldsymbol{A}_1(t)\}u \tag{1}$$

where u is a vector with n components, $\boldsymbol{A}_0$ is a constant matrix of dimension $n \times n$ with negative real eigenvalues and $\boldsymbol{A}_1(t)$ is a random matrix, α is a parameter which measures the strength of fluctuation.

It is convenient to assume that $\boldsymbol{A}_1(t)$ is a stationary process with $\langle \boldsymbol{A}_1(t)\rangle = 0$. Equation (1) sets the two timescales of the system, as measured by the inverse of the largest eigenvalue of the matrix $\boldsymbol{A}_0$ and the timescale of fluctuations of $\boldsymbol{A}_1(t)$ (more precisely the correlation time of fluctuation). In the problem of Brownian motion where one deals with very fast fluctuations such that the correlation time τ_c is essentially the shortest timescale in the dynamics, one thus follows the evolution of the average $\langle u\rangle$ on a coarse-grained timescale.

Before proceeding further we now make two remarks. First, since in this context we have considered a stochastic process in which the fluctuations are weak and adiabatically slow, $\boldsymbol{A}_1(t)$ is an adiabatic stochastic process. Therefore, the usual procedure of systematic expansion in $\alpha\tau_c$ which relies on the smallness of τ_c, is not valid. We thus resort to a different approach based on an expansion in $\alpha|\mu|^{-1}$, where $|\mu|$ refers to the largest eigenvalue of $\boldsymbol{A}_0$ matrix. Second, we *do not* make any *a priori assumption about the nature of the stochastic process*, such as Gaussian or dichotomic etc. The only assumption that has been made about the stochastic process is that the inverse of the damping rate is much shorter than the correlation time of fluctuations $\boldsymbol{A}_1(t)$.

As a first step we introduce an interaction representation as given by,

$$u(t) = \exp(\boldsymbol{A}_0 t)v(t) \tag{2}$$

and applying it to equation (1) we obtain,

$$\dot{v} = \alpha \boldsymbol{V}(t)v \tag{3}$$

where,

$$\boldsymbol{V}(t) = \exp(-\boldsymbol{A}_0 t)\boldsymbol{A}_1(t)\exp(\boldsymbol{A}_0 t). \tag{4}$$

On integration equation (3) yields,

$$v(t) = v(0) + \alpha \int_0^t \boldsymbol{V}(t')v(t')\,\mathrm{d}t'. \tag{5}$$

On iterating equation (5) once, we are led to an ensemble average equation of the form,

$$\langle v(t)\rangle = v(0) + \alpha^2 \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \langle \boldsymbol{V}(t')\boldsymbol{V}(t'')v(t'')\rangle. \tag{6}$$

The equation is still exact since no second-order approximation has been used.

Now taking the time derivative of $v(t)$ we arrive at the following integro-differential equation in which the initial value $v(0)$ no longer appears,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t)\rangle = \alpha^2 \int_0^t \langle \boldsymbol{V}(t)\boldsymbol{V}(t')v(t')\rangle \, \mathrm{d}t'. \tag{7}$$

Making use of a change of integration variable $t' = t - \tau$ and reverting back to the original representation we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \int_0^t \langle \boldsymbol{A}_1(t)\exp(\boldsymbol{A}_0\tau)\boldsymbol{A}_1(t-\tau)u(t-\tau)\rangle \, \mathrm{d}\tau. \tag{8}$$

The *adiabatic following assumption* (see the discussion at the end of this section), that $\boldsymbol{A}_1(t)$ and the components of $u(t)$ vary slowly on the scale of inverse of $\boldsymbol{A}_0$, can now be utilized. Following Crisp [4] we note that a Taylor series expansion of $\boldsymbol{A}_1(t-\tau)u(t-\tau)$ in the average $\langle\cdots\rangle$ of the α^2-term in equation (8) allows us to reduce the above equation to the following form,

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \langle \boldsymbol{A}_1(t)\boldsymbol{I}_n \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}_1(t)u(t)]\rangle. \tag{9}$$

$\boldsymbol{I}_n$ can also be written as

$$\begin{aligned} I_n^{ik} &= \int_0^\infty \mathrm{d}\tau\, \tau^n \sum_j D_{ij} \mathrm{e}^{\mu_{jj}\tau} D_{jk}^{-1} \\ &= \sum_j D_{ij} \frac{n!}{\mu_{jj}^{n+1}} D_{jk}^{-1} \qquad \operatorname{Re}\mu_{jj} < 0. \end{aligned}$$

Equation (9) can then be rewritten in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle + \alpha^2 \sum_{n=o}^{\infty} (-1)^n \langle \boldsymbol{A}_1(t)\boldsymbol{D}\boldsymbol{E}_{n+1}\boldsymbol{D}^{-1} \frac{\mathrm{d}^n}{\mathrm{d}t^n}[\boldsymbol{A}_1(t)u(t)]\rangle \tag{10}$$

where we use

$$\boldsymbol{I}_n = n!\boldsymbol{D}\boldsymbol{E}_{n+1}\boldsymbol{D}^{-1}. \tag{11}$$

Here $\boldsymbol{D}$ is a matrix which diagonalizes $\boldsymbol{A}_0$ and

$$\boldsymbol{E}_{n+1} = \begin{pmatrix} \frac{1}{\mu_{11}^{n+1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\mu_{jj}^{n+1}} \end{pmatrix}$$

and μ_{jj} are the eigenvalues of $\boldsymbol{A}_0$.

Although equation (10) involves an infinite series it is expected to yield useful results in the adiabatic following approximation. Under this approximation the quantity $[\boldsymbol{A}_1(t)u(t)]$ varies very little (such that $\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\boldsymbol{A}_1 u)$ in equation (10) is small) and also since $|\mu_{jj}|$ in $\boldsymbol{E}_{n+1}$ is large the series in equation (10) (which is thus an expansion in $\alpha|\mu|^{-1}$) converges rapidly. Keeping only the two lowest-order terms we arrive at,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \boldsymbol{A}_0\langle u\rangle &+ \alpha^2\langle \boldsymbol{A}_1(t)\boldsymbol{X}_1\boldsymbol{A}_1(t)u(t)\rangle - \alpha^2\langle \boldsymbol{A}_1(t)\boldsymbol{X}_2\dot{\boldsymbol{A}}_1(t)u(t)\rangle \\ &- \alpha^2\langle \boldsymbol{A}_1(t)\boldsymbol{X}_2\boldsymbol{A}_1(t)\dot{u}(t)\rangle \end{aligned} \tag{12}$$

where,

$$X_{n+1} = DE_{n+1}D^{-1}.$$

It is evident that the average $\langle \dot{u} \rangle$ is related to a more complicated average. Following Bourret [10], we now implement the decoupling approximation. This allows us to break up the average as a product of averages. Keeping terms only of the order of α^2 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t)\rangle = \{A_0 + \alpha^2[\langle A_1(t)X_1A_1(t)\rangle - \langle A_1(t)X_2\dot{A}_1(t)\rangle - \langle A_1(t)X_2A_1(t)\rangle A_0]\}\langle u(t)\rangle. \tag{13}$$

Thus the average of $u(t)$ obeys a nonstochastic differential equation in which the effect of weak adiabatic fluctuations is accounted for by renormalizing A_0 through the addition of constant terms of the order of α^2.

The implementation of Bourret's decoupling approximation [10] is a major step for almost any treatment of multiplicative noise up to date [2, 3, 12]. This is because of the fact that the average of a product of stochastic quantities does not factorize into the product of averages, although it has been argued that good approximations can be derived by assuming such a factorization. In the case of fast fluctuations it has been justified if the driving stochastic noise has a short correlation time such that the Kubo number $\alpha^2\tau_c$ is very small in the cummulant expansion scheme (an expansion in $\alpha\tau_c$). The factorization has been shown to be exact in the limit of zero correlation time and in some cases of specific noise processes [3, 12]. The solution for the average can then be found exactly.

In contrast to cummulant expansion (valid in the case of fast fluctuation which relies on an expansion in $\alpha\tau_c$) the present scheme of adiabatic following approximation results in a perturbation series, where the nth term is of order $\alpha\frac{\mathrm{d}^n}{\mathrm{d}t^n}[A(t)u(t)]/\mu_{jj}^{n+1}$ and the convergence of the series is assured since the numerator varies little in the scale of $1/|\mu_{jj}^{n+1}|$. Equation (13) is a result of decoupling approximation employed in this expansion scheme. If one neglects the free motion due to the A_0 term then equation (13), which gives the lowest-order evolution, asserts that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle \sim \frac{\alpha^2}{|\mu|}\langle u\rangle.$$

The contribution of $|\mu|^{-1}$ is derived from X_1 of the first term in equation (13), (i.e. due to the E_{n+1} matrix). Note that because of full integration over τ in moving from equation (8) to equation (9) the correlation time τ_c does not appear in equation (13) and the timescale set by the dynamics is $|\mu|^{-1}$ only. For a fast process on the other hand the counterpart of the last relation is [12]

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle \sim \alpha^2\tau_c\langle u\rangle.$$

It is also easy to calculate the relative error made in the decoupling approximation. We first note that equation (13) is obtained from equation (8). To the second order it means omitting terms of the order $(\alpha\Delta t)^3$ and higher (where Δt is the coarse-grained timescale over which $\langle u\rangle$ evolves). As the lower bound of Δt is determined by $|\mu|^{-1}$, it implies that we neglect terms of the order $(\alpha|\mu|^{-1})^3$ in the evolution equation. The relative error made in the decoupling approximation is thus of the order $(\alpha|\mu|^{-1})^3$ which is well within the order of lowest-order evolution. We thus see that the adiabatic expansion is an expansion in $\alpha|\mu|^{-1}$ and the decoupling approximation in the slow fluctuation is valid where $\alpha^2|\mu|^{-1}$ is very small. Thus $u(t)$ on average (on the right-hand side of equation (12)) is realized as an average $\langle u(t)\rangle$ (which varies in the coarse-grained timescale Δt) in equation (13) pertaining to the separation of the timescales in the inequality (II) in section 1.

Before closing this section a few pertinent points regarding the notion of the '*adiabatic following approximation*' and its genesis may be noted. The notion has acquired special relevance in the quantum optical context where one is concerned with a two-level atom interacting with single-mode electromagnetic field. The model is described by the standard Bloch equations, where the field strength varies slowly on the timescale of the inverse of the damping constant or the frequency detuning between the atom and the field. If the field is varying adiabatically enough, then the population inversion of the Bloch vector components would *follow the field adiabatically* in going from ground to upper state, i.e. the ground-state population is adiabatically inverted. The term '*adiabatic following*' is thus used to describe collectively the associated experimental phenomena [19].

3. Probabilistic considerations: extension to nonlinear equations

We now generalize equation (1) to a stochastic nonlinear differential equation written in the following form

$$\dot{u}_\nu = F_\nu(\{u_\nu\}, t; \xi(t)) \qquad \nu = 1, 2, \ldots, N. \tag{14}$$

The above equation determines a stochastic process with some initial given condition $\{u_\nu(0)\}$. $\xi(t)$ is the adiabatic stochastic process. It may be pointed out that the treatment given in the last section cannot be extended directly to this equation to obtain an equation for average $\langle u\rangle$ since nonlinearity in equation (14) results in higher moments. However, it is possible to transform the nonlinear problem to a linear one if one considers the motion of a representative point u in n-dimensional space $(u_1 \ldots u_n)$ as governed by equation (14). The equation of continuity, which expresses the conservation of points determines the variation of density in time,

$$\frac{\partial\rho(u,t)}{\partial t} = -\sum_\nu \frac{\partial}{\partial u_\nu} F_\nu(\{u_\nu\}, t; \xi(t))\rho(u,t) \tag{15}$$

or more compactly

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\boldsymbol{F}\rho. \tag{16}$$

Equation (15) is a linear stochastic differential equation and is an ideal candidate for the method discussed in section 2 for the linear case. We emphasize here that the basis of this analysis is essentially the two approximations as introduced earlier and *no further approximation* is needed to extend the analysis to the nonlinear domain.

$\boldsymbol{F}$ can now be split as

$$\boldsymbol{F}(\{u_\nu\}, t; \xi(t)) = \boldsymbol{F}_0(\{u_\nu\}) + \alpha\boldsymbol{F}_1(\{u_\nu\}, t; \xi(t)) \tag{17}$$

where $\boldsymbol{F}_0(\{u_\nu\})$ is the constant part and $\boldsymbol{F}_1(\{u_\nu\}, t; \xi(t))$ is the random part with $\langle\boldsymbol{F}_1(t)\rangle = 0$; α is the parameter defined earlier which measures the strength of fluctuation. Equation (16) therefore takes the following form,

$$\dot{\rho}(u,t) = (\boldsymbol{A}_0 + \alpha\boldsymbol{A}_1)\rho(u,t) \tag{18}$$

where $\boldsymbol{A}_0 = -\nabla\cdot\boldsymbol{F}_0$ and $\boldsymbol{A}_1 = -\nabla\cdot\boldsymbol{F}_1$. The symbol ∇ is used for the operator that differentiates everything that comes after it with respect to u.

With the above identification of $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ we are now in a position to apply the fundamental equation (9) derived in the earlier section, to equation (18). We have

$$\frac{\partial}{\partial t} P(u,t) = \left[-\nabla \cdot \boldsymbol{F}_0 P + \alpha^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left\langle -\nabla \cdot \boldsymbol{F}_1 \boldsymbol{I}_n \frac{\mathrm{d}^n}{\mathrm{d}t^n} (-\nabla \cdot \boldsymbol{F}_1 P) \right\rangle \right] \tag{19}$$

where $\langle \rho(u,t) \rangle = P(u,t)$ and also

$$\boldsymbol{I}_n = \int_0^{\infty} \mathrm{d}\tau \, \mathrm{e}^{-\tau \nabla \cdot \boldsymbol{F}_0} \tau^n. \tag{20}$$

The adiabatic following approximation may now be invoked again in the spirit of the earlier treatment in section 2 to obtain

$$\frac{\partial}{\partial t} P(u,t) = -\nabla \cdot [\boldsymbol{F}_0 + \alpha^2 \langle \boldsymbol{F}_1 \boldsymbol{I}_0 \nabla \cdot \boldsymbol{F}_1 \rangle - \alpha^2 \langle \boldsymbol{F}_1 \boldsymbol{I}_1 \nabla \cdot \dot{\boldsymbol{F}}_1 \rangle$$
$$+ \alpha^2 \langle \boldsymbol{F}_1 \boldsymbol{I}_1 \nabla \cdot \boldsymbol{F}_1 \nabla \cdot \boldsymbol{F}_0 \rangle] P(u,t) \tag{21}$$

where we keep terms of the order of α^2 for $n = 0$ and 1 of the series in equation (19).

Our next task is to simplify further the expressions for the averages in equation (21). To this end we first note that the operator $\exp(-\tau \nabla \cdot \boldsymbol{F}_0)$ provides the solution of the equation

$$\frac{\partial f(u,t)}{\partial t} = -\nabla \cdot \boldsymbol{F}_0 f(u,t) \tag{22}$$

(f signifies the unperturbed part of P) which can be found explicitly in terms of characteristic curves. The equation

$$\dot{u} = \boldsymbol{F}_0(u)$$

for fixed t determines a mapping from $u(\tau = 0)$ to $u(\tau)$, i.e. $u \to u^{\tau}$ with inverse $(u^{\tau})^{-\tau} = u$. The solution of equation (22) is

$$f(u,t) = f(u^{-t}, 0) \left| \frac{\mathrm{d}(u^{-t})}{\mathrm{d}(u)} \right| = \mathrm{e}^{-t \nabla \cdot \boldsymbol{F}_0} f(u,0) \tag{23}$$

$\left|\frac{\mathrm{d}(u^{-t})}{\mathrm{d}(u)}\right|$ being a Jacobian determinant. The effect of $\exp(-t\nabla \cdot \boldsymbol{F}_0)$ or $f(u)$ is as follows

$$\exp(-t \nabla \cdot \boldsymbol{F}_0) f(u,0) = f(u^{-t}, 0) \left| \frac{\mathrm{d}(u^{-t})}{\mathrm{d}(u)} \right|. \tag{24}$$

The relation (24) may be used to simplify the average in equation (21). We thus have

$$\langle \nabla \cdot \boldsymbol{F}_1 \boldsymbol{I}_0 \nabla \cdot \boldsymbol{F}_1 \rangle = \nabla \cdot \int_0^{\infty} \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau \tag{25}$$

$$\langle \nabla \cdot \boldsymbol{F}_1 \boldsymbol{I}_1 \nabla \cdot \dot{\boldsymbol{F}}_1 \rangle = \nabla \cdot \int_0^{\infty} \tau \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \dot{\boldsymbol{F}}_1(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau \tag{26}$$

$$\langle \nabla \cdot \boldsymbol{F}_1 \boldsymbol{I}_1 \nabla \cdot \boldsymbol{F}_1 \nabla \cdot \boldsymbol{F}_0 \rangle = \nabla \cdot \int_0^{\infty} \tau \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(u^{-\tau}) \nabla_{-\tau} \cdot \boldsymbol{F}_0(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau. \tag{27}$$

The use of equations (25)–(27) reduces equation (21) to a more tractable form,

$$\frac{\partial}{\partial t} P(u,t) = -\nabla \cdot \left\{ \boldsymbol{F}_0 - \alpha^2 \int_0^{\infty} \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau \right.$$
$$+ \alpha^2 \int_0^{\infty} \tau \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \dot{\boldsymbol{F}}_1(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau$$
$$\left. - \alpha^2 \int_0^{\infty} \tau \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(u^{-\tau}) \nabla_{-\tau} \cdot \boldsymbol{F}_0(u^{-\tau}) \rangle \left| \frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u} \right| \mathrm{d}\tau \right\} P(u,t) \tag{28}$$

where $\nabla_{-\tau}$ denotes the differential with respect to $u_{-\tau}$. Equation (28) is our basic result in this section. The equation is second order in α, i.e. of the order of $\alpha^2|\mu|^{-1}$, where $|\mu|$ refers to the eigenvalue of $\boldsymbol{A}_0$. In our earlier communication [5] we have shown the convergence of the series in $\alpha|\mu|^{-1}$, pertaining to the separation of the timescales implied in (II) in section 1. We also remark that it is possible to extend the treatment to higher order, in general. Note also that the equation involves three differentiation of $P(u, t)$ with respect to the components of u and is a third-order equation. The appearance of third-order noise beyond the usual Fokker–Planck terms is a characteristic of the process we consider here. We discuss this aspect in more detail in the following two sections.

4. Adiabatic stochasticity in phase space

We now consider the motion of a particle in one dimension subject to a force $K(x)$ depending on the position x, a frictional force $-\beta\dot{x}$ and a stochastic force $\alpha\xi(t)$. Here β is a measure of the damping of the system and α is the strength of adiabatically slow fluctuations $\xi(t)$. We thus write

$$m\ddot{x} + \beta\dot{x} = K(x) + \alpha\xi(t). \tag{29}$$

The corresponding problem of fast fluctuation $\alpha\xi(t)$ was studied by Kramers [13] as a model of simple chemical reactions and by Bixon and Zwanzig [14] as a model for fluctuating nonlinear systems.

For simplicity we set $m = 1$ and $\dot{x} = v$. Then the two components of u in this example are x and v. Taking equation (17) into account we have

$$\begin{aligned} &F_{0x} = v \qquad F_{1x} = 0 \\ &F_{0v} = -\beta v + K(x) \qquad F_{1v} = \alpha\xi(t). \end{aligned} \tag{30}$$

By considering a small variation of v in time τ, one obtains (from the unperturbed version of equation (29)) the Jacobian determinant for the 'unperturbed' mapping $u \to u^{\tau}$

$$\left|\frac{\mathrm{d}u^{-\tau}}{\mathrm{d}u}\right| \equiv \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| = 1 + \beta\tau + \mathcal{O}(\tau^2) \tag{31}$$

and

$$\begin{aligned} \frac{\partial}{\partial v^{-\tau}} &= (1 - \beta\tau)\frac{\partial}{\partial v} + \tau\frac{\partial}{\partial x} + \mathcal{O}(\tau^2) \\ \frac{\partial}{\partial x^{-\tau}} &= \frac{\partial}{\partial x} + \tau\frac{\partial K(x)}{\partial x}\frac{\partial}{\partial v} + \mathcal{O}(\tau^2). \end{aligned} \tag{32}$$

Equation (28) now reduces to the following form

$$\begin{aligned} \frac{\partial}{\partial t}P(x, v, t) = -\nabla\cdot\Bigg\{ &\boldsymbol{F}_0 - \alpha^2\int_0^\infty \langle \boldsymbol{F}_1\nabla_{-\tau}\cdot\boldsymbol{F}_1(x^{-\tau}, v^{-\tau})\rangle \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau \\ &+\alpha^2\int_0^\infty \tau\langle \boldsymbol{F}_1\nabla_{-\tau}\cdot\dot{\boldsymbol{F}}_1(x^{-\tau}, v^{-\tau})\rangle \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau \\ &-\alpha^2\int_0^\infty \tau\langle \boldsymbol{F}_1\nabla_{-\tau}\cdot\boldsymbol{F}_1(x^{-\tau}, v^{-\tau})\nabla_{-\tau}\cdot\boldsymbol{F}_0(x^{-\tau}, v^{-\tau})\rangle \\ &\times\left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau\Bigg\} P(x, v, t). \end{aligned} \tag{33}$$

Making use of relations (30)–(32) one may reduce the terms on the right-hand side of equation (33) to more simplified forms. Thus

$$-\nabla \cdot \boldsymbol{F}_0 P(x, v, t) = -v\frac{\partial P}{\partial x} + \beta\frac{\partial}{\partial v}(vP) - K(x)\frac{\partial P}{\partial v} \tag{34}$$

$$\alpha^2\nabla \cdot \int_0^\infty \langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(x^{-\tau}, v^{-\tau})\rangle \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau\, P(x, v, t) = \alpha^2\tilde{c}_0\frac{\partial^2 P}{\partial v^2} + \alpha^2\tilde{c}_1\frac{\partial^2 P}{\partial v\partial x} \tag{35}$$

$$\alpha^2\nabla \cdot \int_0^\infty \tau\langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \dot{\boldsymbol{F}}_1(x^{-\tau}, v^{-\tau})\rangle \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau\, P(x, v, t) = -\alpha^2\tilde{c}_2\frac{\partial^2 P}{\partial v^2} \tag{36}$$

$$\begin{aligned}\alpha^2\nabla \cdot \int_0^\infty \tau\langle \boldsymbol{F}_1 \nabla_{-\tau} \cdot \boldsymbol{F}_1(x^{-\tau}, v^{-\tau})\nabla_{-\tau} \cdot \boldsymbol{F}_0(x^{-\tau}, v^{-\tau})\rangle \left|\frac{\mathrm{d}(x^{-\tau}, v^{-\tau})}{\mathrm{d}(x, v)}\right| \mathrm{d}\tau\, P(x, v, t)\\ = \alpha^2\tilde{c}_1\left[2\frac{\partial^2 P}{\partial v\partial x} + v\frac{\partial^3 P}{\partial v^2\partial x} + K(x)\frac{\partial^3 P}{\partial v^3} - \beta\frac{\partial^3}{\partial v^3}(vP)\right]\end{aligned} \tag{37}$$

where

$$\begin{aligned}\tilde{c}_0 &= \int_0^\infty \langle\xi(t)\xi(t-\tau)\rangle\,\mathrm{d}\tau\\ \tilde{c}_1 &= \int_0^\infty \tau\langle\xi(t)\xi(t-\tau)\rangle\,\mathrm{d}\tau\\ \tilde{c}_2 &= \int_0^\infty \tau\langle\xi(t)\dot{\xi}(t-\tau)\rangle\,\mathrm{d}\tau.\end{aligned} \tag{38}$$

The final equation for the average motion corresponding to an adiabatic stochastic evolution in phase space is,

$$\begin{aligned}\frac{\partial}{\partial t}P(x, v, t) = -v\frac{\partial P}{\partial x} + \beta\frac{\partial}{\partial v}(vP) - K(x)\frac{\partial P}{\partial v} + \alpha^2(\tilde{c}_0 - \tilde{c}_2)\frac{\partial^2 P}{\partial v^2} + 3\alpha^2\tilde{c}_1\frac{\partial^2 P}{\partial v\partial x}\\ +\alpha^2\tilde{c}_1\left[v\frac{\partial^3 P}{\partial v^2\partial x} + K(x)\frac{\partial^3 P}{\partial v^3} - \beta\frac{\partial^3}{\partial v^3}(vP)\right].\end{aligned} \tag{39}$$

The remarkable departure from the standard form of the Fokker–Planck equation is thus evident in equation (39) since it contains third derivative terms. The magnitude of their contribution is dependent on how much 'unperturbed' x and v vary during τ which is of the order of $|\mu|^{-1}$. We also point out that in the above derivation Bourret's decoupling approximation [10] has been used as in the treatment of linear equation in section 2.

5. Adiabatic fluctuations in velocity space

We now consider the motion of a particle with velocity v in the presence of fluctuations $\alpha\xi(t)$ which is adiabatically slow. The equation of motion is given by

$$\dot{v} = -\beta v + \alpha\xi(t). \tag{40}$$

The corresponding problem of a Brownian particle with fast fluctuations is a century-old problem in physical science, in general. Following the procedure described in the earlier section we first identify the perturbed and the unperturbed part of $\boldsymbol{F}$, i.e.

$$\boldsymbol{F}_0 = -\beta v \qquad \boldsymbol{F}_1 = \alpha\xi(t) \tag{41}$$

and calculate the Jacobian $|\frac{dv^{-\tau}}{dv}|$ for the mapping $v \to v^{\tau}$ for the 'unperturbed' motion

$$\left|\frac{dv^{-\tau}}{dv}\right| = e^{\beta\tau}. \tag{42}$$

Also note that

$$\nabla_{-\tau} = e^{-\beta\tau}\frac{\partial}{\partial v}. \tag{43}$$

The evolution of the probability distribution function $P(v, t)$ is then given by (terms of the order α^2)

$$\frac{\partial}{\partial t}P(v, t) = \beta\frac{\partial}{\partial v}(vP) + \alpha^2 c_{12}\frac{\partial^2 P}{\partial v^2} - \alpha^2\beta c_3\frac{\partial^3}{\partial v^3}(vP) \tag{44}$$

where

$$\begin{aligned} c_{12} &= c_1 - c_2 \\ c_1 &= \int_0^\infty \langle\xi(t)\xi(t-\tau)\rangle\, d\tau \\ c_2 &= \int_0^\infty \tau\langle\xi(t)\dot{\xi}(t-\tau)\rangle\, d\tau \\ c_3 &= \int_0^\infty \tau\langle\xi(t)\xi(t-\tau)\rangle\, d\tau. \end{aligned} \tag{45}$$

While in the absence of the third term, the first two terms on the right-hand side of equation (44) correspond to drift and diffusion terms in the Fokker–Planck description of an Ornstein–Uhlenbeck process, the third-derivative term precludes the possibility of any straightforward interpretation of the equation. Similar equations with third-order noise, although not very common, may be encountered [7] however, in the treatment of trimolecular reactions and also in quantum optics describing optical bistability in terms of the associated Wigner distribution function for the reduced density operator in symmetrical ordering for the radiation field.

We now return to equation (44) which after some modification becomes

$$\frac{\partial}{\partial t}P(v, t) = \beta\frac{\partial}{\partial v}[vP(v, t)] + D_1\frac{\partial^2 P(v, t)}{\partial v^2} - \beta D_2 v\frac{\partial^3 P(v, t)}{\partial v^3} \tag{46}$$

where

$$\begin{aligned} D_1 &= \alpha^2(c_{12} - 3c_3\beta) \\ D_2 &= \alpha^2 c_3. \end{aligned} \tag{47}$$

We now transform equation (46) to Fourier space by defining the conditional probability $P(v, t|v_0, 0)$ and its Fourier transform as

$$P(v, t|v_0, 0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dk\, e^{ikv}\tilde{P}(k, t|v_0, 0) \tag{48}$$

to obtain

$$\frac{\partial}{\partial t}\tilde{P}(k, t|v_0, 0) = -\beta(k + D_2k^3)\frac{\partial\tilde{P}}{\partial k} - (D_1 + 3\beta D_2)k^2\tilde{P}. \tag{49}$$

The linear partial differential equation (49) can be solved by the method of characteristics. For the initial condition (at time $t = 0$)

$$P(v, 0|v_0, 0) = \delta(v - v_0) \tag{50}$$

the solution is

$$\tilde{P}(k,t|v_0,0) = \frac{1}{(1+Bk^2)^A} \exp\left[-\mathrm{i}kv_0\sqrt{\frac{f(t)}{1+Bk^2}}\right] \tag{51}$$

where

$$\begin{aligned} f(t) &= \mathrm{e}^{-2\beta t} \\ A &= c_{12}/2\beta c_3 \\ B &= \alpha^2 c_3\{1-f(t)\}. \end{aligned} \tag{52}$$

It is easy to check that equation (51) satisfies

$$\tilde{P}^*(k,t|v_0,0) = \tilde{P}(-k,t|v_0,0)$$

and the characteristic function (51) is of stable form.

The conditional probability density $P(v,t|v_0,0)$ is obtained by inverse Fourier transformation of equation (51) and is given by,

$$P(v,t|v_0,0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \frac{1}{(1+Bk^2)^A} \exp\left[\mathrm{i}kv - \mathrm{i}kv_0\sqrt{\frac{f(t)}{1+Bk^2}}\right]. \tag{53}$$

It is evident that $P(v,t|v_0,0)$ is a Fourier transform of a stable characteristic function. Hence the solution (53) forms a stable distribution in the variable v. Equation (53) is one of the most important results of this paper.

Although an explicit expression for $P(v,t|v_0,0)$ is difficult to obtain, closed-form solutions for $P(v,t|v_0,0)$ for stationary state can be easily obtained. In the long-time limit the characteristic function (51) reduces to its asymptotic form

$$\tilde{P}(k,\infty) = \frac{1}{(1+D_2k^2)^A} \tag{54}$$

which results in a steady-state distribution of stable form. Explicitly for small A, i.e. large β this is given by

$$P_{\mathrm{st}}(v) = \frac{|v|^{A+1}}{2^A D_2^{\frac{A}{2}}\Gamma(A)v^2}\mathrm{e}^{-\frac{|v|}{\sqrt{D_2}}}. \tag{55}$$

It is interesting to note that the dominant behaviour of $P_{\mathrm{st}}(v)$ for small v. This is given by a power law of the form

$$\begin{aligned} P_{\mathrm{st}}(v) &\sim \frac{|v|^{A+1}}{v^2} \\ &\sim |v|^{-1+A}. \end{aligned} \tag{56}$$

Such power law behaviour is also apparent for Levy processes [9, 11] but for the large v regime.

Although an explicit solution for the probability distribution $P(v,t|v_0,0)$ is difficult to obtain for arbitrary time, a few statistical properties of the process can, however, be obtained from the calculation of variances and higher moments. For convenience, we define such moments by subtracting the mean motion of the variables, i.e. we calculate the moments of $\Delta v(= v - v_0\mathrm{e}^{-\beta t})$. Thus we write

$$\langle|\Delta v|^m\rangle = \int_{-\infty}^{+\infty} (v - v_0\mathrm{e}^{-\beta t})^m P(v,t|v_0,0) \tag{57}$$

or more explicitly

$$\langle|\Delta v|^m\rangle = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}k \frac{1}{(1+Bk^2)^A} \exp[\mathrm{i}kv_0\mathrm{e}^{-\beta t}\{1-(1+Bk^2)^{-\frac{1}{2}}\}] \times \int_{-\infty}^{+\infty} \mathrm{d}\Delta v|\Delta v|^m \mathrm{e}^{\mathrm{i}k\Delta v}. \tag{58}$$

After some algebra we obtain

$$\langle|\Delta v|^m\rangle = (-\mathrm{i})^m \int_{-\infty}^{+\infty} \mathrm{d}k \frac{1}{(1+Bk^2)^A} \exp[\mathrm{i}kv_0\mathrm{e}^{-\beta t}\{1-(1+Bk^2)^{-\frac{1}{2}}\}] \frac{\partial^m \delta(k)}{\partial k^m}. \tag{59}$$

In principle, using the property of Dirac δ-function and appropriate integrations any moment can be calculated from the above relation. We quote the results explicitly for the first three moments

$$\begin{aligned} &\text{for } m = 1 \langle|\Delta v|\rangle = 0 \\ &\text{for } m = 2 \langle|\Delta v|^2\rangle = \frac{\alpha^2 c_{12}}{\beta}(1-\mathrm{e}^{-2\beta t}) \\ &\text{for } m = 3 \langle|\Delta v|^3\rangle = 3\alpha^2 c_3 v_0 \mathrm{e}^{-\beta t}(1-\mathrm{e}^{-2\beta t}). \end{aligned} \tag{60}$$

It is thus evident that unlike Levy processes [9, 11] the moments are finite.

We thus observe that because of the linear dissipation β, a system driven by adiabatic noise reaches a steady state which is stable. However, since the noise is of external origin, the outward flow of energy due to linear dissipation need not always balance the inward flow of energy supplied by the adiabatic fluctuations and hence a fluctuation–dissipation relation cannot be conceived in this case.

6. Conclusions

In conclusion, we consider herein a dynamical system driven by external adiabatic fluctuations. Based on the ‘adiabatic following approximation’ we have made a systematic separation of timescales to carry out an expansion in $\alpha|\mu|^{-1}$ to obtain a linear differential equation for the average solution, where α is the strength of fluctuation and $|\mu|$ is the largest eigenvalue of the unperturbed system. The main results of this study can be summarized as follows.

(i) The probability distribution functions obey the differential equations of motion which contain third-order terms beyond the usual Fokker–Planck terms. The adiabatic fluctuations thus may give rise to non-Gaussian noise.

(ii) We have examined in detail the corresponding equation in velocity space and the characteristic function is shown to obey a simple third-order differential equation which can be solved exactly in closed form. The characteristic function is found to be of stable form.

(iii) Although third-order noise, in general, leads to serious interpretative difficulties in terms of truly probabilistic considerations in several cases, we show that in this problem of adiabatic stochasticity in velocity space, statistical properties of the processes are more transparent. It is of special interest to note that in contrast to Levy processes all the variances and higher moments are finite and the probability distribution is of stable form.

(iv) Because of linear dissipation, the system driven by adiabatic fluctuations reaches a stable steady state.

(v) For small arguments the probability distribution function obeys a power law behaviour which is reminiscent of Levy processes.

The stochastification by adding rapid fluctuating terms has been applied earlier to a wide variety of physical problems described by linear relaxation equations [15], hydrodynamic equations [16], Maxwell equations in a medium [17], Boltzmann equation [18] etc. Our analysis shows that the present method might reveal interesting consequences in such cases where the added fluctuating terms in question are adiabatically slow. We hope to address such issues in future communications.

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